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Constructive Interpolation of Infinite Sequences by Entire Functions

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Abstract: The study is motivated by a theoretical question raised by Warming and Hyett in a famous publication on the *Modified Equation Approach* [1] in which they assumed the existence of a function that interpolates numerical data defined over a uniform grid, and that can be expanded, over an indefinite domain, in Taylor's series of the independent variables x and t. We establish constructively that the problem of interpolation of an arbitrary infinite sequence of data by an *entire* function of x (and possibly t) admits uncountably many solutions. In the case of a single variable, if the data are bounded, the interpolant can be made bounded, and all its derivatives bounded. Besides, the construction is generalized to the interpolation of the values of the function and its derivatives up to an arbitrarily prescribed order (*Hermitian interpolation*). The proposed interpolant depends on a free parameter λ, and its behavior as λ varies is illustrated in a particular case by numerical experiment.

Key-words: interpolation, finite-difference grid, entire function, absolutely-convergent series, constructive proof

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Interpolation constructive des suites infinies par des fonctions entières

Résumé : Cette étude est motivée par une question théorique soulevée par Warming et Hyett dans une publication célèbre sur l'Equation Equivalente [1] dans laquelle ils ont fait l'hypothèse de l'existence d'une fonction qui interpole des données numériques définies sur un maillage uniforme et développable, sur un domaine indéfini, en série de Taylor des variables indépendantes x et t. On montre constructivement que le problème de l'interpolation d'une suite infinie quelconque de données par une fonction *entière* de x (et éventuellement de t) admet une infinité non-dénombrable de solutions. Dans le cas d'une seule variable, si les données sont bornées, l'interpolant peut être construit borné, ainsi que ses dérivées. En outre, la construction se généralise à l'interpolation des valeurs de la fonction et de ses dérivées jusqu'à un ordre prescrit arbitrairement (*interpolation hermitienne*). L'interpolant proposé dépend d'un paramètre réel λ, et son comportement quand λ varie est illustré par expérience numérique dans un cas particulier.

Mots-clés : interpolation, maillage différences-finies, fonction entière, série absolument convergente, preuve constructive

Prologue

At the time I was in my twenties, every young French man of sound mind and body had to find an excuse, or "serve the Nation" for one year in the military. Having found no excuse, I served. Thanks, however, to my background in Aeronautics, I was quickly directed to the French Air Force Academy. However, the irony of the situation was that because of the very small contingent of English teachers that year, and my daring invocation of my Doctorate in "Philosophy" (with a major in Aerospace!), I managed, after all the technical positions of instructors had already been assigned, to avoid the duties of a common soldier and was directed instead to the language department.

Teaching English was a very interesting experience, at least for me, and probably my most intensive period of learning grammar. Nevertheless, military life alternates assignments with long periods of boring free time, and I used to keep my mind busy with a number of "casses-têtes chinois" (Chinese head breakers): puzzles, mostly of a mathematical nature, of my own construction.

Most of the findings reported here (Lemma 4.1, Corollary 4.1 and part of Theorem 4.1) were established at that time, although it is only recently, almost by accident, that I completed these findings with the simple and constructive result on boundedness (Theorem 3.1) and convinced myself to gather them together in a publication.

These puzzles have entertained my reveries from time to time during long hikes in the French Alps. I hope that some readers will also find the subject entertaining, and if not, perhaps somewhat useful to their own constructions.

1 Introduction

For boundary-value problems, involving a Partial-Differential Equation (PDE), typically of elliptic type and defined in short by

$$A\,u = f \qquad (+\text{ boundary conditions}) \tag{1}$$

the accuracy of a finite-difference scheme,

$$A_h\,u_h = f_h \qquad (+\text{ discrete boundary conditions}) \tag{2}$$

is classically analyzed by first evaluating the *trunction error*,

$$\theta_h = A_h\,u - f_h \tag{3}$$

obtained by injecting the formal solution, $u(x)$ ($x \in \Omega \subset \mathbb{R}^d$), to the continuous (PDE) problem, (1), into the finite-difference equation, (2), and using the *a priori* known regularity of the solution $u(x)$ to expand the quantity in a truncated Taylor's series of the mesh parameters

$$\Delta x = (\Delta x_1,\, \Delta x_2,\, ...,\, \Delta x_d) \tag{4}$$

In simple cases, this yields an information of the type :

$$\|\theta_h\| \le C_u\, h^\alpha \tag{5}$$

(in the sense of some norm) where the constant C_u involves the bound on a derivative of $u(x)$,

$$h = \|\Delta x\| \tag{6}$$

and α is the order of accuracy of the numerical approximation (often 2). In test problems where (1) is a well-posed elliptic problem, if the domain $\Omega = [0,1]^d$, A_h is often a positive-definite matrix whose inverse admits a known bound μ. Consequently, the *approximation error*,

$$e_h = u - u_h = A_h^{-1}\,\theta_h \tag{7}$$

admits a simple bound,

$$\|e_h\| \le \mu\,\|\theta_h\| \le \mu C_u\, h^\alpha \tag{8}$$

Thus in these simple cases, the knowledge of (information on) the solution $u(x)$ to the continuous problem suffices to establish such convergence bound.

For time-dependent problems, of hyperbolic, parabolic or mixed type,

$$u_t = \mathcal{L}\,u \quad (+\text{ initial, and possibly boundary conditions}) \tag{9}$$

(where the operator $\mathcal{L}$ is not necessarily linear), the situation is not so straightforward and the accuracy analysis is usually done somewhat differently. In the early seventies, several authors proposed theoretical and practical methods for the analysis of finite-difference schemes, mostly in the perspective of solving hyperbolic problems of fluid dynamics. In particular Lerat & Peyret [2] introduced the concept of *équation équivalente* (equivalent equation) and Warming & Hyett [1] the very similar concept of *modified equation.* To outline these concepts, let us consider for simplicity, the practical case where (9) is the linear, hyperbolic, advection (or wave) equation :

$$u_t + c\,u_x = 0 \quad (+\text{ initial conditions}) \tag{10}$$

and the domain $\Omega_{x,t} = \mathbb{R} \times \mathbb{R}^+$ for a pure-initial value problem. A possible numerical scheme for (10) is obtained by approximating u_t by a first-order forward finite-difference in time, and u_x by a first-order backward finite-difference in space (which is stable for $c > 0$) :

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + c\,\frac{u_j^n - u_{j-1}^n}{\Delta x} = 0 \tag{11}$$

where a uniform mesh has been assumed ($x_j = j\Delta x\,;\ t^n = n\Delta t$). Equation (11) provides a recurrence formula giving the unknowns at time level $n+1$ explicitly in terms of the analogous data at time level n. The accuracy analysis is here conducted in four steps :

1. Step 1 :
   Let $u(x,t)$ be the solution to the PDE to be approximated, here (10); define an interpolant $\tilde{u}(x,t)$ of the numerical data $\{u_j^n\}$ produced by the finite-difference scheme, here (11).

2. Step 2 :
   Assuming sufficient regularity of the *unknown interpolant* $\tilde{u}$, the finite-difference equation, (11), reformulated as
   $$\frac{\tilde{u}(x_j,t^{n+1})-\tilde{u}(x_j,t^n)}{\Delta t}+c\,\frac{\tilde{u}(x_j,t^n)-\tilde{u}(x_{j-1},t^n)}{\Delta x}=0 \tag{12}$$
   is expanded into two power series of Δt and Δx respectively, yielding the following equation satisfied by $\tilde{u}$ at meshpoints only :
   $$\tilde{u}_t+\frac{\Delta t}{2}\,\tilde{u}_{tt}+\frac{\Delta t^2}{6}\,\tilde{u}_{ttt}+\cdots+c\left[\tilde{u}_x-\frac{\Delta x}{2}\,\tilde{u}_{xx}+\frac{\Delta x^2}{6}\,\tilde{u}_{xxx}+\ldots\right]=0\quad(\text{at all }(x,t)=(x_j,t^n)) \tag{13}$$

3. Step 3 :
   Consequences of (13) are derived and injected into (13) itself to eliminate recursively the time derivatives of order greater than 1. For this, we introduce the notation
   $$\tau_k(\Delta t,\Delta x) \tag{14}$$
   for a series (alternately a finite expansion) in powers of Δt and Δx whose coefficients are proportional to derivatives of $\tilde{u}(x,t)$ (x and t being additional variables), and whose lowest-order terms are of order k. Then we observe that such an expansion can be differentiated term by term w.r.t. x or t. For example, (13) implies that :
   $$\tilde{u}_t=-c\,\tilde{u}_x+\tau_1(\Delta t,\Delta x)\,,\ \ \tau_1=-\frac{\Delta t}{2}\,\tilde{u}_{tt}+c\,\frac{\Delta x}{2}\,\tilde{u}_{xx}+\ldots \tag{15}$$
   and consequently, differentiating w.r.t. the coefficients of the series maintains its general form :
   $$\begin{aligned}
   \tilde{u}_{tt}&=-c\tilde{u}_{xt}+\tau_1'(\Delta t,\Delta x) & \tau_1'&=\left.\tau_1\right)_t & (16)\\
   &=-c\,\tilde{u}_{tx}+\tau_1'(\Delta t,\Delta x) & &=-\frac{\Delta t}{2}\,\tilde{u}_{ttt}+c\,\frac{\Delta x}{2}\,\tilde{u}_{xxt}+\ldots & (17)\\
   &=-c\,\left[-c\,\tilde{u}_x+\tau_1(\Delta t,\Delta x)\right]_x+\tau_1'(\Delta t,\Delta x) & \left.\tau_1\right)_x&=-\frac{\Delta t}{2}\,\tilde{u}_{ttx}+c\,\frac{\Delta x}{2}\,\tilde{u}_{xxx}+\ldots & (18)\\
   &=c^2\,\tilde{u}_{xx}+\tau_1''(\Delta t,\Delta x) & \tau_1''&=-c\ \left.\tau_1\right)_x+\tau_1' & (19)
   \end{aligned}$$
   and injecting this result into (13) yields the following equation which now replaces (13) :
   $$\tilde{u}_t+c\,\tilde{u}_x=\frac{c}{2}\,(\Delta x-c\Delta t)\,\tilde{u}_{xx}+\tau_2(\Delta t,\Delta x) \tag{20}$$
   This equation provides an expression of $\tilde{u}_t$ in terms of space derivatives that is valid up to order 1. Thus $\tilde{u}_{tt}$ and $\tilde{u}_{ttt}$ can also be expressed in the same way, and substituting these new expressions into (20) would yield the second-order term, and so on.

   Remark : In this process, a PDE is simplified recursively by using consequences of the PDE itself, and *not* consequences of the original PDE, here (10). Earlier works that failed to make this observation led to expressions of the truncation error terms improprer beyond the principal one.

If this process is continued indefinetely, one obtains a formal equation in which on the left-hand side appears the operator that serves to define the PDE under consideration, and on the right-hand side, the truncation errors expressed in terms of space derivatives only.

When conducting this elimination process up to order k, in the terminology of Peyret & Lerat, one obtains the PDE equivalent to the finite-difference scheme up to order k (– a concept of finite expansion, valid under *smoothness* assumptions only –).

When the process is completed to eliminate *all* the space derivatives, one obtains the so-called *modified equation* according to the Warming & Hyett terminology, which in the specific case considered here has the following form :

$$\tilde{u}_t + c\,\tilde{u}_x = P_1(\Delta t, \Delta x)\,\tilde{u}_{xx} + P_2(\Delta t, \Delta x)\,\tilde{u}_{xxx} + \dots \tag{21}$$

where P_k $(k = 1,\ 2,\ \dots)$ is a homogeneous polynomial of degree k in Δt, Δx (see [1] for the precise expression); e.g. $P_1(\Delta t, \Delta x) = \frac{1}{2}c\,(\Delta x - c\Delta t)$.

Note that the latter concept is related to the convergence of two (infinite) Taylor's series, whose justification requires more than smoothness, namely *analyticity* (with radii of convergence in the t and x directions uniformely sufficiently large in relation to the size of the finite-difference molecule). Warming & Hyett indicated how such equation could be derived formally in a rather general setting; they originally developed a computer program in the MACSYMA formal language to perform this automatically; they also indicated how this equation can serve to analyze the accuracy and to some extent, the stability of the finite-difference scheme. Before returning to this important question, observe that (21) has only been established at meshpoints, and let us indicate the final step in the analysis :

4. Step 4 :
   A rigorous analysis should include a converse result (not really established in either [2] or [1] to the author's understanding) according which one would consider the PDE :

$$v_t + c\,v_x = P_1(\Delta t, \Delta x)\,v_{xx} + P_2(\Delta t, \Delta x)\,v_{xxx} + \dots \tag{22}$$

   and establish well-posedness, and using implications in the reverse direction, prove that :

$$\forall j\ ,\ \forall n\ :\ v(x_j, t^n) = \tilde{u}(x_j, t^n) = u_j^n \tag{23}$$

   This final step is essential to guarantee that (22) can be solved as a standard PDE, yielding an information concerning globally the numerical integration.

The validity of the modified equation approach has been questioned by certain authors (e.g. [3]). To the author's opinion, it is clearly justified for two-time-level schemes applied to pure initial-value problems, but requires specific additional justifications when the context is more general.

Let us assume that all steps, including Step 4, have been carried over successfully. Then, the interest of either the *equivalent* or *modified* equation relies in the fact that the effect of the truncation error terms is visible by inspection of the right-hand side of (22) : in the particular case considered above, one concludes that the scheme is exact when $c\Delta t = \Delta x$ (all the coefficients P_k vanish)[1], first-order accurate otherwise; the principal truncation-error term is proportional to the second derivative v_{xx} : it is a dissipation term provided the coefficient is positive, yielding the well-known necessary stability condition

$$\nu = \frac{c\Delta t}{\Delta x} \le 1 \tag{24}$$

[1] This is not surprising since the scheme then reduces to $u_j^{n+1} = u_{j-1}^n$ which is an exact transport of information along a segment of characteristics.

where ν is the Courant number.

For a second-order scheme the principal truncation error term would involve the third derivative v_{xxx}. In such case, the truncation error is said to be primarily *dispersive* and it affects mostly phase errors in a Fourier analysis.

Clearly, the approach is potentially capable of providing qualitative and quantitative information concerning numerical methods applied to time-dependent simple test or more complex equations. It is thus a very useful theoretical and practical tool of analysis, used extensively in the literature.

This report is motivated by the following observation : even though the interpolant $\tilde{u}(x,t)$ is abandoned at Step 4, the development in Step 2 would be more rigorous if one could guarantee the existence of an interpolant *of sufficient regularity* without making assumptions on the infinite sequence $\{u_j^n\}$. Warming and Hyett [1] themselves have said : *"... we assume the existence of a continuously differentiable function $u(x,t)$ [$\tilde{u}(x,t)$ above] which coincides at the mesh points – at least in some local sense – with the exact solution of the difference equation ..."*. Note that it is trivial to construct an interpolant only constrained to C^∞ regularity : it suffices for this to superimpose bell-shaped functions having this regularity and small disjoint compact supports, centered at the meshpoints and sized to the values to be interpolated locally, that is :

$$\tilde{u}(x,t) = \sum_j \sum_n u_j^n \, \varphi(\frac{x}{\Delta x} - j)\, \varphi(\frac{t}{\Delta t} - n) \tag{25}$$

where $\varphi(s)$ can be, for example, the function of class $\mathsf{C}^\infty(\mathbb{R})$ equal to $\exp[1/(4s^2-1)+1]$ over the open interval $]-\frac{1}{2},\frac{1}{2}[$ and 0 outside. The function $\varphi(s)$ is *precisely not analytic* at the endpoints of its support, and in a sense, the above interpolant is *not uniformly regular* between meshpoints, and thus presents little interest. What is really needed, in our understanding of the *modified equation* approach, is *analyticity* in both x and t, at each meshpoint, with the additional constraint that the corresponding Taylor expansions admit radii of convergence uniformly larger than dimensions related to the finite-difference molecule. Additionally, a *global*, rather than *local*, existence result would be preferable.

2 Problem statement and report organization

Taking now some distance with finite-difference schemes, and considering first real-valued functions of the single real variable x, we consider the following questions : given an arbitrary infinite sequence of real numbers $\{u_n\}$ ($n \in \mathbb{Z}$; $u_n \in \mathbb{R}$), and a real strictly positive number R, among all the smooth interpolants $u(x)$ satisfying :

$$\forall n \in \mathbb{Z} : \; u(n) = u_n \tag{26}$$

exists there any that is analytic over the real line ? If yes, are there any such that the radius of convergence of the series expressing $u(x)$ locally is uniformly greater than the specified constant R ? Exists there any that is entire (i.e. the same with R infinite) ?

This report provides a constructive proof that all three above questions admit positive answers regardless the sequence $\{u_n\}$, and the set of the corresponding interpolants is uncountable. Furthermore, if the sequence $\{u_n\}$ is bounded, entire interpolants $u(x)$ can be constructed to satisfy uniformly the same bound, and such that all their derivatives are uniformly bounded over the real line.

The proposed proofs are constructive in the sense that they provide formulas that can be used to calculate possible interpolants numerically given the sequence $\{u_n\}$. The construction is defined by : either (78) or (94) for a bounded sequence; (105)-(107) for a general sequence; (109)-(118)-(121) in case the derivatives up to a specified order p must also be interpolated. Finally the construction is extended to the case of a doubly-indexed sequence $\{ u_j^n \}$ to provide interpolants that are entire in x (at all fixed time t) and in t (at all fixed abscissa x); in this case the construction is given by (125).

Some of these results could undoubtedly be proved more directly by application of known theorems of the theory of functional approximation; in particular the boundiness result of the next section, see subsequently. We publicize our proof because it is simple and constructive and it relies only on elementary notions, principally the summability of absolutely convergent series, and can thus be understood, and possibly modified to adjust to slightly different situations, by a large community of practitioners of numerical analysis.

The report is organized as follows : in Section 3, we prove that any sequence can be bounded by the sequence of the values at integer abscissas of some entire function; in Section 4 we prove our main interpolation result concerning bounded sequences, and we generalize to an arbitrary sequence; then, in Section 5 we examine two extensions : the extension to the interpolation of derivatives as well as function values (*Hermitian interpolation*), and the extension to the interpolation of data by functions entire in several variables; finally in Section 6, the proposed interpolants are illustrated in a particular case to demonstrate the effect of a free parameter in the construction, and we conclude by stating a number of related open questions of interest.

Before entering technical details, we point out that except specifically mentioned otherwise, all the functions considered here are real-valued functions of the real variable.

3 Bounding sequences

In this section, we prove the following

Theorem 3.1 (*Bounding an infinite sequence by the values of an entire function)*
Let $\{u_n\}$ ($n \in \mathbb{Z}$; $u_n \in \mathbb{R}$) be an arbitrary infinite sequence of real numbers. Uncountably many real-valued entire functions $\mu(x)$ of the real variable x are such that :

$$\forall n \in \mathbb{Z} \, : \, |u_n| \leq \mu(n) \tag{27}$$

Two proofs are provided : one is based on a known result from the theory of functional approximation; the second employs an independent construction.

3.1 Proof based on a known theorem

Recall the following known theorem :

Theorem 3.2 (*(from [4]), § 12.11, pp. 248-249)*
A continuous function can be approximated arbitrarily closely on $(-\infty, \infty)$ by entire functions; in fact,

$$sup \, |f(x) - g(x)| \, \varphi(|x|) \tag{28}$$

can be made arbitrarily small, with $g(x)$ entire, no matter how fast the given function $\varphi(x)$ grows.

Before using this theorem, let us observe that it states a very strong result on the possibility to approximate a continuous function, here $f(x)$, by an entire function $g(x)$ since the faster the growth of the free function $\varphi(x)$, the more stringent the condition on the supremum is.

Admitting this theorem, define $f(x)$ to be the following continuous piecewise linear function :

$$f(x) = |u_n| + (x - n)(|u_{n+1}| - |u_n|) \quad \forall x \in (n, n+1) \, , \, \forall n \in \mathbb{Z} \tag{29}$$

Choose $\varphi(x) = 1$ and let $g(x)$ be an entire function for which the above sup is at most equal to 1. Then, the uncountable set of entire functions $\mu(x) = g(x) + 1 + \phi(x)^2$, where $\phi(x)$ is an arbitrary entire function, satisfy all the requirements of Theorem 3.1. □

3.2 Alternate constructive proof of Theorem 3.1

Let $\{u_n\}$ $(n \in \mathbb{Z}\,,\ u_n \in \mathbb{R})$ be an arbitrary infinite sequence of real numbers. We will first consider the problem of bounding the subsequence corresponding to the positive integer values of n $(n \in \mathbb{N})$).

Define the sequence $\{v_n\}$ $(n \in \mathbb{N})$ as follows :

$$v_n = \max\Big(|u_n|, (1+\epsilon_1)\,\ln(n+\epsilon_2)\Big) \tag{30}$$

where ϵ_1 and ϵ_2 are two strictly-positive small numbers. In this way :

$$\forall n \in \mathbb{N}\ :\ v_n \geq |u_n| \geq 0 \tag{31}$$

and for large n :

$$v_n\, e^{-v_n} \leq \frac{(1+\epsilon_1)\,\ln(n+\epsilon_2)}{(n+\epsilon_2)^{1+\epsilon_1}} \leq \frac{1}{(n+\epsilon_2)^{1+(\epsilon_1/2)}} \tag{32}$$

so that the numerical series

$$\sum_{n=0}^{\infty} v_n\, e^{-v_n} \tag{33}$$

converges, and the series of functions

$$\sum_{n=0}^{\infty} v_n\, e^{-v_n (x-n)^2} \tag{34}$$

also does for all $x \in \mathbb{R}$. Define :

$$\mu_+(x) = \sum_{n=0}^{\infty} v_n\, e^{-v_n (x-n)^2} \tag{35}$$

It follows immediately that :

$$\forall k \in \mathbb{N}\ :\ \mu_+(k) \geq v_k \geq |u_k| \tag{36}$$

To complete the part of the proof concerning $\mu_+(x)$, it remains to establish that this function is entire. For this, first observe that each in term in (35) is an entire function and can be expanded as follows :

$$\begin{aligned}
v_n\, e^{-v_n (x-n)^2} &= v_n\, e^{-n^2 v_n}\ e^{-v_n (x^2-2xn)} \\
&= v_n\, e^{-n^2 v_n}\ \sum_{k=0}^{\infty} \frac{v_n^k}{k!}\,(2xn-x^2)^k \\
&= v_n\, e^{-n^2 v_n}\ \sum_{k=0}^{\infty} v_n^k \sum_{\ell=0}^{k} \frac{(2nx)^\ell\,(-x^2)^{k-\ell}}{\ell!\,(k-\ell)!} \\
&= v_n\, e^{-n^2 v_n}\ \sum_{k=0}^{\infty} v_n^k \sum_{\ell=0}^{k} \frac{(-1)^{k-\ell}\,(2n)^\ell\,x^{2k-\ell}}{\ell!\,(k-\ell)!} \\
&= v_n\, e^{-n^2 v_n}\ \sum_{m=0}^{\infty} \alpha_{n,m}\,x^m
\end{aligned} \tag{37}$$

where the coefficients $\{\alpha_{n,m}\}$ are defined by

$$\alpha_{n,m} = \sum_{\substack{k \in \mathbb{N}/\\ (m/2) \leq k \leq m}} \frac{v_n^k\,(-1)^{m-k}\,(2n)^{2k-m}}{(2k-m)!\,(m-k)!} \tag{38}$$

so that :

$$\mu_+(x) = \sum_{n=0}^{\infty} v_n \, e^{-n^2 v_n} \sum_{m=0}^{\infty} \alpha_{n,m} \, x^m \tag{39}$$

Now, it is obvious that :

$$\forall n, m \; : \; |\alpha_{n,m}| \leq \beta_{n,m} \tag{40}$$

where :

$$\beta_{n,m} = \sum_{\substack{k \in \mathbb{N} / \\ (m/2) \leq k \leq m}} \frac{v_n^k \, (2n)^{2k-m}}{(2k-m)! \, (m-k)!} \tag{41}$$

and clearly, for any fixed $x \in \mathbb{R}$:

$$\sum_{m=0}^{\infty} |\alpha_{n,m} \, x^m| \leq \sum_{m=0}^{\infty} \beta_{n,m} \, |x|^m = e^{v_n (x^2 + 2n|x|)} \tag{42}$$

Consider an integer $N \geq |x| + \sqrt{2x^2 + 1}$, so that :

$$\forall n \geq N \; : \; n^2 - x^2 - 2n|x| \geq 1 \tag{43}$$

It follows that :

$$\sum_{n=0}^{N-1} \sum_{m=0}^{\infty} |v_n \, e^{-n^2 v_n} \, \alpha_{n,m} \, x^m| < \infty \tag{44}$$

since the terms being added are the absolute values of the terms appearing in the power series expansions expressing the values at x of a finite number of entire functions; besides :

$$\sum_{n=N}^{\infty} \sum_{m=0}^{\infty} |v_n \, e^{-n^2 v_n} \, \alpha_{n,m} \, x^m| \leq \sum_{n=N}^{\infty} v_n \, e^{-n^2 v_n} \, e^{v_n (x^2 + 2n|x|)} \leq \sum_{n=N}^{\infty} v_n \, e^{-v_n} < \infty \tag{45}$$

Consequently, the series of doubly-indexed terms in (39) is absolutely convergent for any fixed x. It is therefore legitimate to permute the summation signs, and get the new expression

$$\mu_+(x) = \sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} \alpha_{n,m} \, v_n \, e^{-n^2 v_n} \right) x^m \tag{46}$$

which indicates that the function $\mu_+(x)$ is entire, since this is valid for arbitrarily large, but fixed x.

Using a similar construction, we can identify a uniformly-positive entire function $\mu_-(x)$ such that :

$$\forall n \in \mathbb{N} \; : \; \mu_-(n) \geq |u_{-n}| \tag{47}$$

Evidently, the uncountable set of entire functions

$$\mu(x) = \mu_+(x) + \mu_-(-x) + \phi(x)^2 \tag{48}$$

where $\phi(x)$ is an arbitrary entire function, meets all the requirements of Theorem 3.1, whose proof is now complete. □

4 Interpolating infinite sequences

4.1 Preliminaries

Consider the entire function

$$\boxed{\varphi(x) = \frac{\sin\, \pi x}{\pi x} = \sum_{n=0}^{\infty} \frac{(-1)^n\, (\pi x)^{2n}}{(2n+1)!}} \tag{49}$$

This function is uniformly bounded :

$$\forall x \in \mathbb{R}\ :\ |\,\varphi(x)\,| \leq 1 \tag{50}$$

The m-th derivative is calculated according to Leibniz's rule :

$$\varphi^{(m)}(x) = \sum_{k=0}^{m} C_m^k\ (\sin \pi x)^{(m-k)} \left(\frac{1}{\pi x}\right)^{(k)} = \sum_{k=0}^{m} C_m^k\, \pi^{m-k}\, \sin\left(\pi x + (m-k)\frac{\pi}{2}\right) \frac{(-1)^k\, k!}{\pi\, x^{k+1}} \tag{51}$$

where $C_m^k = m!/[k!\,(m-k)!]$; hence for $|x| \geq 1$:

$$|\,\varphi^{(m)}(x)\,| \leq \frac{m!}{\pi} \sum_{k=0}^{m} \frac{\pi^{m-k}}{(m-k)!} \leq \frac{e^{\pi}}{\pi}\, m! \tag{52}$$

which proves that the m-th derivative is uniformly bounded over the real line. Lastly,

$$\forall n \in \mathbb{Z}\,,\ \varphi(n) = \delta_{n,0} \tag{53}$$

($= 1$ if $n = 0$, and 0 otherwise).

Consider now the function

$$\boxed{\psi_\lambda(x) = e^{-\lambda\, x^2}\, \varphi(x)} \tag{54}$$

where λ is a strictly-positive real parameter. This function is entire and uniformly bounded by 1. Applying again Leibniz's rule to calculate the m-th derivative gives :

$$\psi_\lambda^{(m)}(x) = \sum_{k=0}^{m} C_m^k\, \frac{d^{m-k}}{dx^{m-k}} \left(e^{-\lambda\, x^2}\right) \frac{d^k}{dx^k}\Big(\varphi(x)\Big) = e^{-\lambda\, x^2} \sum_{k=0}^{m} C_m^k\, P_{m-k}(x)\, \varphi^{(k)}(x) \tag{55}$$

in which the sequence of polynomials $\{\, P_n(x)\,\}$ has been defined by[2]

$$\forall n \in \mathbb{N},\ \ P_n(x) = e^{\lambda\, x^2}\, \frac{d^n}{dx^n} \left(e^{-\lambda\, x^2}\right) \tag{56}$$

Equation (55) indicates that all the derivatives of $\psi_\lambda(x)$ are uniformly bounded.

Finally, note that the sequence of functions :

$$h_m(x) = e^{\frac{\lambda x^2}{2}}\, \psi_\lambda^{(m)}(x) \tag{57}$$

are entire, vanish at ∞, and admit uniform bounds, denoted for convenience

$$H_m = \max_{x \in \mathbb{R}}\, |\, h_m(x)\,| \tag{58}$$

[2]This generating formula indicates that the polynomials $\{\, P_n(x)\,\}$ are related to the Hermite polynomials $\{\, H_n(x)\,\}$ by the relation : $P_n(x) = (-1)^n\, \lambda^{n/2}\, H_n(x\sqrt{\lambda})$ (Rodrigues' formula).

4.2 Basic Lemma and Corollary for bounded sequences

Lemma 4.1 (*Basic Lemma for bounded sequences*)
Let $\{ u_n \}$ ($n \in \mathbb{Z}$; $u_n \in \mathbb{R}$) be a bounded infinite sequence of real numbers. Among all the smooth interpolants $u(x)$ satisfying :

$$\forall n \in \mathbb{Z} : \; u(n) = u_n \tag{59}$$

uncountably many are entire, uniformly bounded, and admit derivatives that are uniformly bounded over the real line.

Proof : Let A be an upper bound on $|\, u_n \,|$ and consider the function $\psi_\lambda(x)$ defined in the preliminaries, and some natural integer m. Then :

$$\forall n \in \mathbb{Z}\,,\; \forall x \in \mathbb{R} : \; |\, u_n \, \psi_\lambda^{(m)}(x-n) \,| = |\, u_n \, e^{-\frac{\lambda}{2}(x-n)^2} \, h_m(x-n) \,| \leq A \, H_m \, e^{-\frac{\lambda}{2}(x-n)^2} \tag{60}$$

Consequently, for any $m \in \mathbb{N}$, the functional series of general term $u_n \, \psi_\lambda^{(m)}(x-n)$ is absolutely convergent. Thus define :

$$u_\lambda^{(m)}(x) = \sum_{n\in\mathbb{Z}} u_n \, \psi_\lambda^{(m)}(x-n) = \sum_{n\in\mathbb{Z}} u_n \, e^{-\frac{\lambda}{2}(x-n)^2} \, h_m(x-n) \tag{61}$$

where for the moment, m (over u_λ) is only understood as a superscript. Additionally :

$$\forall x \in \mathbb{R} : \; |\, u_\lambda^{(m)}(x) \,| \leq A \, H_m \, \rho(x) \tag{62}$$

where $\rho(x)$ is the following positive function :

$$\rho(x) = \sum_{n\in\mathbb{Z}} e^{-\frac{\lambda}{2}(x-n)^2} \tag{63}$$

From

$$\rho(x) = \sum_{n\leq 0} e^{-\frac{\lambda}{2}(x-n)^2} + \sum_{n>0} e^{-\frac{\lambda}{2}(x-n)^2} \tag{64}$$

it follows that for $x \in [0,1]$:

$$0 < \rho(x) \leq \sum_{n\leq 0} e^{-\frac{\lambda}{2}n^2} + \sum_{n\geq 0} e^{-\frac{\lambda}{2}n^2} = 2 \sum_{n=0}^{\infty} e^{-\frac{\lambda}{2}n^2} = B_\lambda \tag{65}$$

But since the function $\rho(x)$ is 1-periodic, the above bound holds uniformly over the real line. Consequently,

$$\forall m \in \mathbb{N}, \; \forall x \in \mathbb{R} : \; |\, u_\lambda^{(m)}(x) \,| \leq A \, B_\lambda \, H_m \tag{66}$$

Now, we are going to prove that the superscript m over $u_\lambda(x)$ in (61) is indeed an order of differentiation.

For this let x_0 be an arbitrary but fixed real number and let us first consider the derivative to the right of x_0. Let n_0 be the largest integer less or equal to x_0, and let

$$\sigma = \frac{u_\lambda^{(m)}(x) - u_\lambda^{(m)}(x_0)}{x - x_0} \tag{67}$$

It follows that :

$$\begin{aligned}
\sigma &= \sum_{n\in\mathbb{Z}} u_n \, \frac{\psi_\lambda^{(m)}(x-n) - \psi_\lambda^{(m)}(x_0-n)}{x-x_0} \\
&= \sum_{n\in\mathbb{Z}} u_n \, \left[\psi_\lambda^{(m+1)}(x_0-n) + (x-x_0)\, \psi_\lambda^{(m+2)}(\xi_n - n) \right] \\
&= u_\lambda^{(m+1)}(x_0) + (x-x_0)\, \tilde{h}(x)
\end{aligned} \tag{68}$$

where $\xi_n \in]x_0, x[\subset]n_0, n_0+1[$ and

$$\tilde{h}(x) = \sum_{n\in\mathbb{Z}} u_n \, \psi_\lambda^{(m+2)}(\xi_n - n) = \sum_{n\in\mathbb{Z}} u_n \, e^{-\frac{\lambda}{2}(\xi_n - n)^2} \, h_{m+2}(\xi_n - n) \tag{69}$$

in which $h_{m+2}(x)$ is an entire function uniformly bounded by the constant H_{m+2}. Hence, the following bound applies to $\tilde{h}(x)$:

$$\forall x \in \mathbb{R} \,:\, |\,\tilde{h}(x)\,| \leq A\, H_{m+2} \sum_{n\in\mathbb{Z}} e^{-\frac{\lambda}{2}(\xi_n - n)^2} \tag{70}$$

But,

$$\sum_{n\in\mathbb{Z}} e^{-\frac{\lambda}{2}(\xi_n - n)^2} = \sum_{n\leq n_0} e^{-\frac{\lambda}{2}(\xi_n - n)^2} + \sum_{n\geq n_0+1} e^{-\frac{\lambda}{2}(\xi_n - n)^2} \tag{71}$$

and since for all $n \in \mathbb{Z}$, $n_0 < \xi_n < n_0 + 1$, it follows that :

$$\sum_{n\in\mathbb{Z}} e^{-\frac{\lambda}{2}(\xi_n - n)^2} \leq 2 \sum_{k=0}^{\infty} e^{-\frac{\lambda}{2}k^2} = B_\lambda \tag{72}$$

and thus :

$$\forall x \in \mathbb{R} \,:\, |\,\tilde{h}(x)\,| \leq A\, B_\lambda \, H_{m+2} \tag{73}$$

and consequently :

$$\lim_{x\to x_0^+} \sigma = u_\lambda^{(m+1)}(x_0) \tag{74}$$

and the same could be proved for the limit $x \to x_0^-$. Since x_0 is arbitrary, it follows that :

$$\frac{d}{dx} u_\lambda^{(m)} = u_\lambda^{(m+1)} \tag{75}$$

identically, and letting

$$u_\lambda^{(0)} = u_\lambda \tag{76}$$

we finally conclude that :

$$\forall m \in \mathbb{N} \,:\, u_\lambda^{(m)} = \frac{d^m}{dx^m} u_\lambda \tag{77}$$

At this stage, we have established that the function u_λ is infinitely differentiable and all its derivatives are uniformly bounded over the real line. We are now going to prove that it is entire. For this, note that :

$$\boxed{u_\lambda(x) = \sum_{n\in\mathbb{Z}} u_n \, e^{-\lambda(x-n)^2} \, \varphi(x-n) = e^{-\lambda x^2} \, v_\lambda(x)} \tag{78}$$

where :

$$v_\lambda(x) = \sum_{n\in\mathbb{Z}} u_n \, e^{-\lambda n^2} \, e^{2\lambda n x} \, \varphi(x-n) \tag{79}$$

We can now use known power series expansions :

$$
\begin{aligned}
e^{2\lambda n x}\,\varphi(x-n) &= e^{2\lambda n x}\,\frac{\sin\,\pi(x-n)}{\pi(x-n)} \\
&= \sum_{q=0}^{\infty}\frac{(2\lambda n x)^q}{q!}\sum_{r=0}^{\infty}(-1)^r\,\frac{[\pi(x-n)]^{2r}}{(2r+1)!} \\
&= \sum_{q=0}^{\infty}\frac{(2\lambda n x)^q}{q!}\sum_{r=0}^{\infty}\sum_{t=0}^{2r}\frac{(-1)^r\pi^{2r}}{(2r+1)!}\,C_{2r}^{t}\,x^t\,(-n)^{2r-t} \\
&= \sum_{p=0}^{\infty} a_{n,p}\,x^p
\end{aligned}
\tag{80}
$$

where the coefficients $\{\,a_{n,p}\,\}$ are given by :

$$
a_{n,p} = \sum_{\substack{(q,r,t)\in\mathbb{N}^3/\\ q+t=p,\\ t\le 2r}}\frac{(2\lambda n)^q}{q!}\,\frac{(-1)^r\pi^{2r}}{(2r+1)!}\,C_{2r}^{t}\,(-n)^{2r-t}
\tag{81}
$$

Consequently :

$$
\begin{aligned}
\sum_{p=0}^{\infty}|a_{n,p}\,x^p| &= \sum_{p=0}^{\infty}\left|\sum_{\substack{(q,r,t)\in\mathbb{N}^3/\\ q+t=p,\\ t\le 2r}}\frac{(2\lambda n x)^q}{q!}\,\frac{(-1)^r\pi^{2r}}{(2r+1)!}\,C_{2r}^{t}\,x^t\,(-n)^{2r-t}\right| \\
&\le \underbrace{\sum_{p=0}^{\infty}\sum_{\substack{(q,r,t)\in\mathbb{N}^3/\\ q+t=p,\\ t\le 2r}}\frac{|2\lambda n x|^q}{q!}\,\frac{\pi^{2r}}{(2r+1)!}\,C_{2r}^{t}\,|x|^t\,|n|^{2r-t}}_{e^{2|nx|}\,\frac{\sinh\,[\pi(|x|+|n|)]}{\pi(|x|+|n|)}} \\
&\le a\,e^{b|n|}
\end{aligned}
\tag{82}
$$

where a and b are, for fixed x, positive constants. Consequently :

$$
\sum_{n\in\mathbb{Z}}\sum_{p=0}^{\infty}\left|u_n\,e^{-\lambda n^2}\,a_{n,p}\,x^p\right| \le \sum_{n\in\mathbb{Z}} A\,e^{-\lambda n^2}\,a\,e^{b|n|} < \infty
\tag{83}
$$

Therefore, the series defining the function $v_\lambda(x)$, namely :

$$
v_\lambda(x) = \sum_{n\in\mathbb{Z}}\sum_{p=0}^{\infty} u_n\,e^{-\lambda n^2}\,a_{n,p}\,x^p
\tag{84}
$$

is absolutely convergent, which legitimates permuting the order of the summation signs and concluding :

$$
v_\lambda(x) = \sum_{p=0}^{\infty}\left(\sum_{n\in\mathbb{Z}} u_n\,e^{-\lambda n^2}\,a_{n,p}\right)x^p
\tag{85}
$$

which proves that the function is indeed entire.

Hence, the function $u_\lambda(x)$ in (78) is entire, uniformly bounded and all its derivatives are uniformly bounded over the real line.

Finally, this function satisfies the interpolation condition since for all $k \in \mathbb{Z}$:

$$u_\lambda(k) = \sum_{n\in\mathbb{Z}} u_n \, e^{-\lambda(k-n)^2} \underbrace{\varphi(k-n)}_{\delta_{k-n,0}} = u_k \tag{86}$$

In conclusion, the function $u_\lambda(x)$ meets all the requirements of Lemma 4.1, and since the parameter λ can assume all the values of $\mathbb{R}_+^*$, we have constructed an uncountable infinity of solutions $\{ u_\lambda \}$ to the interpolation problem having the desired regularity. $\square$

From a practical point of view, one would want to select a value of λ sufficiently large to avoid oscillations of large amplitude between interpolated values. Besides, a too large value of λ is also undesirable because it would disconnect interpolated points by intervals where the interpolant would practically vanish. We will show in Section 6 for a specific case, that the sup-norm of the derivative of the interpolant u_λ explodes in both limits $\lambda \to 0^+$ and $\lambda \to \infty$. Hence, intuitively, we expect this norm to be minimum for a specific value of the parameter λ. Thus our construction leads to interpolants whose derivatives are not controlled in norm. However, the following corollary indicates that we can at least maintain the function values of the interpolant within the bounds of the given sequence $\{ u_n \}$.

Corollary 4.1 (*On bounded interpolants of a bounded sequence*)
Let $\{ u_n \}$ ($n \in \mathbb{Z}$; $u_n \in \mathbb{R}$) be a bounded infinite sequence of real numbers satisfying :

$$\forall n \in \mathbb{Z} : \; -1 \le u_n \le 1 \tag{87}$$

Among all the smooth interpolants $u(x)$ satisfying :

$$\forall n \in \mathbb{Z} : \; u(n) = u_n \tag{88}$$

uncountably many are entire, satisfy

$$\forall x \in \mathbb{R} \setminus \mathbb{Z} : \; -1 < u(x) < 1 \tag{89}$$

and admit derivatives that are uniformly bounded over the real line.

Proof : Let us first treat the case of a constant sequence $u_n = C$. Let ϵ be a real strictly-positive free parameter. The set of functions $\{ u_\epsilon \}$ defined, as ϵ varies in $\mathbb{R}_+^*$, by :

$$u_\epsilon(x) = \begin{cases} C\,(1+\epsilon\,\cos^2 \pi x)/(1+\epsilon) & \text{if } C \neq 0 \\ \sin \pi x\,(1+\epsilon\,\cos^2 \pi x)/(1+\epsilon) & \text{if } C = 0 \end{cases} \tag{90}$$

is uncountable and meets all the requirements.

Consider now the general case of a non-constant sequence and define the new sequence :

$$v_n = \sin^{-1} u_n \tag{91}$$

This sequence is bounded and by virtue of Corollary 4.1 there exists an uncountable set V of entire functions $v(x)$ that are bounded, admit bounded derivatives, and satisfy the condition :

$$\forall n \in \mathbb{Z} : \; v(n) = v_n \quad (\forall v \in V) \tag{92}$$

Consider the following application :

$$\Phi \,:\, v \in V \longrightarrow u_v = \Phi(v) \tag{93}$$

where the function u_v is given by :

$$\boxed{u_v(x) = \frac{1+\epsilon\cos^2 \pi x}{1+\epsilon}\,\sin\left[v(x)\right]} \tag{94}$$

in which ϵ is a small, strictly-positive, free parameter. This function is entire, such that

$$\begin{cases} \forall n \in \mathbb{Z} \,:\, u_v(n) = u_n \\ \forall x \in \mathbb{R} \setminus \mathbb{Z} \,:\, -1 < u_v(x) < 1 \end{cases} \tag{95}$$

and has bounded derivatives; hence, it meets all the requirements. The set $\Phi(V)$ is thus a set of solutions to the considered interpolation problem. It remains to prove that it is uncountable. Since the set V is itself uncountable, it suffices for this to establish that the application Φ is injective. For this consider two distinct elements v and w of V, and the difference $d = \Phi(v) - \Phi(w)$. The function $d(x)$ has the following form :

$$d(x) = p(x)\,q(x)\,r(x) \tag{96}$$

where :

$$\begin{cases} p(x) = 2\,\dfrac{1+\epsilon\cos^2 \pi x}{1+\epsilon} \\ q(x) = \sin\dfrac{v(x)-w(x)}{2} \\ r(x) = \cos\dfrac{v(x)+w(x)}{2} \end{cases} \tag{97}$$

The function $p(x)$ is uniformly strictly positive. Let $S(q)$ and $S(r)$ be the sets of zeros of the functions q and r respectively. Let us show that these sets are countable. Recall that the zeros of a nonzero analytic function are countable, which reduces the question to proving that $q(x)$ and $r(x)$ are not identically zero. But, $q(x) = 0$ identically iff :

$$\forall x \in \mathbb{R} \,:\, \frac{v(x)-w(x)}{2} = k\pi \tag{98}$$

for some integer k, constant by continuity. But since the functions $v(x)$ and $w(x)$ interpolate the same sequence $\{\, v_n \,\}$, it follows that $k = 0$, implying that $v = w$, which conflicts with the hypothesis. Hence $q(x)$ is not identically zero, and $S(q)$ is countable. Concerning $S(r)$, a similar reason holds. The condition $r(x) = 0$ holds identically iff

$$\forall x \in \mathbb{R} \,:\, \frac{v(x)+w(x)}{2} = \frac{\pi}{2} + k\pi \tag{99}$$

for some integer k, constant by continuity. Letting $x = n$ $(n \in \mathbb{Z})$ in the above equation gives :

$$\forall n \in \mathbb{Z} \,:\, v(n) = w(n) = v_n = \frac{\pi}{2} + k\pi \tag{100}$$

and consequently

$$\forall n \in \mathbb{Z} \,:\, u_n = \sin v_n = (-1)^k \tag{101}$$

which conflicts with the hypothesis according which the sequence $\{\, u_n \,\}$ is not constant. Hence $r(x)$ is not identically zero, and $S(r)$ is countable.

Finally, the set of zeros of the function $d(x)$,

$$S(d) = S(q) \cup S(r) \tag{102}$$

is countable; hence $d = \Phi(v) - \Phi(w)$ is nonzero, and this proves that the application Φ is indeed injective. The set of interpolants $\Phi(V)$ is therefore uncountable. □

4.3 Interpolating a general sequence

Theorem 4.1 (*Interpolating a general infinite sequence*)
Let $\{ u_n \}$ $(n \in \mathbb{Z}\,;\ u_n \in \mathbb{R})$ *be an arbitrary infinite sequence of real numbers. Uncountably many pairs of entire functions* $\Big(a(x)\,,\ b(x)\Big)$ *are such that :*

$$\forall n \in \mathbb{Z}:\ \ \min\Big(a(n)\,,\ b(n)\Big) \le u_n \le \max\Big(a(n)\,,\ b(n)\Big) \tag{103}$$

Additionally, for any such pair for which $a(x)$ *and* $b(x)$ *are not identical, among all the smooth interpolants* $u(x)$ *satisfying :*

$$\forall n \in \mathbb{Z}:\ u(n) = u_n \tag{104}$$

uncountably many are of the form :

$$\boxed{u(x) = \frac{b(x)+a(x)}{2} + \frac{b(x)-a(x)}{2}\,\theta(x)} \tag{105}$$

where the function $\theta(x)$ *is entire, uniformly bounded, admits uniformly bounded derivatives over the real line, and satisfies :*

$$\begin{cases} \forall n \in \mathbb{Z}:\ \theta(n) = \theta_n \\ \forall x \in \mathbb{R} \setminus \mathbb{Z}:\ \ -1 < \theta(x) < 1 \end{cases} \tag{106}$$

where the sequence $\{ \theta_n \}$ *has been defined by :*

$$\theta_n = \begin{cases} \left[u_n - \dfrac{b(n)+a(n)}{2}\right] \Big/ \left[\dfrac{b(n)-a(n)}{2}\right] & \text{whenever } a(n) \neq b(n) \\ \text{any number in [-1,1], otherwise} \end{cases} \tag{107}$$

Proof : The existence of such pairs of functions $\Big(a(x)\,,\ b(x)\Big)$ is an immediate consequence of Theorem 3.1, since a possible choice is given by $a(x) = -\mu(x)$, $b(x) = +\mu(x)$. After defining the sequence $\{ \theta_n \}$ as indicated, it follows that :

$$\forall n \in \mathbb{Z}:\ \ -1 \le \theta_n \le 1 \tag{108}$$

and the existence of uncountably many functions $\theta(x)$ having the desired properties results directly from applying Corollary 4.1. Finally, to distinct functions $\theta(x)$ correspond distinct interpolants $u(x)$ since the functions a and b being entire (thus analytic) and distinct, the equation $a(x) = b(x)$ admits at most countably many solutions. The set of interpolants $u(x)$ of such type is therefore uncountable. □

5 Extensions

5.1 Hermitian interpolation

In this subsection, we examine the possibility of interpolating derivatives as well as function values.

Lemma 5.1 (*Matching the derivative of order p*)
Let p *be a strictly-positive integer,* $\theta_p(x)$ *a function of class* $\mathrm{C}^\infty(\mathbb{R})$*, and :*

$$\boxed{v_p(x) = \Big(\sin\,\pi x\Big)^p\,\theta_p(x)} \tag{109}$$

Then :

$$\forall n \in \mathbb{Z} \; : \; \begin{cases} v_p(n) = v_p'(n) = \cdots = v_p^{(p-1)}(n) = 0 \\ v_p^{(p)}(n) = (-1)^{np}\, \pi^p \, p!\, \theta_p(n) \end{cases} \tag{110}$$

Proof : by recurrence on p.

For $p = 1$, $v_1(x) = \sin \pi x\, \theta_p(x)$; indeed we have, $\forall n \in \mathbb{Z} \, : \, v_1(n) = 0$ and $v_1'(n) = \pi \cos \pi n\, \theta_p(n) + \sin \pi n\, \theta_p'(n) = (-1)^n \, \pi\, \theta_p(n)$.

Now, assume (110) is true up to some order p, and consider the function $v_{p+1}(x)$. Since :

$$v_{p+1}(x) = \sin \pi x \, v_p(x) \tag{111}$$

Leibniz's rule yields the following, for any positive integer α :

$$v_{p+1}^{(\alpha)}(n) = \sum_{\beta=0}^{\alpha} C_\alpha^\beta \left(\sin \pi x \right)^{(\alpha-\beta)}_{|x=n} v_p^{(\beta)}(n) \tag{112}$$

which can be simplified using the recurrence hypothesis according which :

$$\forall n \in \mathbb{Z}\, , \, \forall \beta \le p-1 \, : \, v_p^{(\beta)}(n) = 0 \tag{113}$$

Two cases are examined :

<u>1st case : $\alpha \le p$.</u> All the terms in (112) corresponding to $\beta \le \alpha - 1 \le p-1$ are equal to 0 because $v_p^{(\beta)}(n) = 0$. The last term corresponds to $\beta = \alpha$; it is therefore multiplied by $\sin \pi n = 0$. Consequently :

$$\forall \alpha \le p \, : \, v_{p+1}^{(\alpha)}(n) = 0 \tag{114}$$

<u>2nd case : $\alpha = p + 1$.</u> Equation (112) writes :

$$\begin{aligned} v_{p+1}^{(p+1)}(n) &= 0 + \underbrace{C_{p+1}^p}_{p+1} \underbrace{\left(\sin \pi x \right)'_{|x=n}}_{\pi\,(-1)^n} \underbrace{v_p^{(p)}(n)}_{\substack{(-1)^{np}\, \pi^p\, p!\, \theta_p(n) \\ \text{(by recurrence hypothesis)}}} + C_{p+1}^{p+1} \underbrace{\sin \pi n}_{0} \, v_p^{(p+1)}(n) \\ &= (-1)^{n(p+1)} \, \pi^{p+1} \, (p+1)!\, \theta_p(n) \end{aligned} \tag{115}$$

Equations (114) and (115) are equivalent to the statement of the lemma at order $p+1$, which completes the proof by recurrence on p. □

The above lemma will now serve to prove the following

Theorem 5.1 (*Hermitian interpolation)*
Let p be a positive integer, and $\{ u_n^k \}$ ($n \in \mathbb{Z}\, , \; k \in \mathbb{N}, \; k \le p\, , \; u_n^k \in \mathbb{R}$) an arbitrary infinite doubly-indexed sequence of real numbers. Among all the smooth interpolants $u(x)$ satisfying :

$$\forall n \in \mathbb{Z}\, , \, \forall k \in \mathbb{N} \text{ such that } k \le p \, : \, u^{(k)}(n) = u_n^k \tag{116}$$

uncountably many are entire.

Proof : consider the following proposition which depends on a dummy positive integer α :
$\mathrm{P}(\alpha)$: *uncountably many entire functions $u_\alpha(x)$ satisfy the following :*

$$\forall n \in \mathbb{Z}\,,\ \forall k \in \mathbb{N} \textit{ such that } k \leq \alpha \ :\ u_\alpha^{(k)}(n) = u_n^k \tag{117}$$

We have to prove that $\mathrm{P}(p)$ is true, and we proceed by recurrence on p.

For $p = 0$, $\mathrm{P}(0)$ is true by virtue of Theorem 4.1.

For $p \geq 1$, assume that $\mathrm{P}(\alpha)$ is true for $\alpha = 0, 1, \dots, p-1$ and denote by $u_{p-1}(x)$ any particular interpolant satisfying (117) for $\alpha = p-1$. Let :

$$\boxed{u_p(x) = u_{p-1}(x) + v_p(x)} \tag{118}$$

where the function $v_p(x)$ has the same formal expression as in Lemma 5.1.

Then, using (110) for $k \leq p-1$ yields :

$$u_p^{(k)}(n) = u_{p-1}^{(k)}(n) + v_p^{(k)}(n) = u_n^k + 0 \quad (\forall n \in \mathbb{Z}) \tag{119}$$

and for $k = p$:

$$u_p^{(p)}(n) = u_{p-1}^{(p)}(n) + v_p^{(p)}(n) = u_{p-1}^{(p)}(n) + (-1)^{np}\,\pi^p\,p!\,\theta_p(n) = u_n^p \quad (\forall n \in \mathbb{Z}) \tag{120}$$

provided the following condition is imposed on the adjustable function $\theta_p(x)$:

$$\boxed{\theta_p(n) = \frac{u_n^p - u_{p-1}^{(p)}(n)}{(-1)^{np}\,\pi^p\,p!} \quad (\forall n \in \mathbb{Z})} \tag{121}$$

By virtue of Theorem 4.1, uncountably many entire functions $\theta_p(x)$ satisfy this condition, which completes the proof of $\mathrm{P}(p)$, and by consequence, the recurrence proof. □

5.2 Interpolation in several variables

Altough this study was motived in the introduction by considerations related to functions of two or more variables, our results so far concern functions of a single variable only. In this subsection we indicate how can these results be extended straightforwardly to two variables, and evidently to several.

Thus consider the question of interpolating a general infinite doubly-indexed sequence $\{\,u_j^n\,\}$ ($j \in \mathbb{Z}\,,\ n \in \mathbb{Z}\,,\ u_j^n \in \mathbb{R}$) of real numbers.

We begin by bounding the sequence by the values of a function entire in both x and t. For this define the following sequence :

$$v_j^n = \max\left(|u_j^n|, (1+\epsilon_1)\ln\left(|j| + \epsilon_2\right) \times (1+\epsilon_1')\ln\left(|n| + \epsilon_2'\right)\right) \tag{122}$$

where ϵ_1, ϵ_2, ϵ_1' and ϵ_2' are strictly-positive small real numbers. Let

$$\mu(x,t) = \sum_{j\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} v_j^n\, e^{-v_j^n\left[\lambda(x-j)^2 + \lambda'(n-t)^2\right]} \tag{123}$$

where (λ, λ') is a pair of strictly-positive real numbers. This function is entire in both x and t separately, and :

$$\forall (j,n) \in \mathbb{Z} \times \mathbb{Z} \, : \; |u_j^n| \leq \mu(j,n) \tag{124}$$

Then we propose the following class of interpolants of the sequence $\{ u_j^n \}$:

$$\boxed{u_{\lambda,\lambda'}(x,t) = \mu(x,t) \sum_{j\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \frac{u_j^n}{\mu(j,n)} \, e^{-\lambda(x-j)^2 - \lambda'(t-n)^2} \, \varphi(x-j) \, \varphi(t-n)} \tag{125}$$

which are entire in both x and t separately, for all pairs of strictly-positive real parameters (λ, λ'), thus proving the existence of uncountably many solutions. □

This result can be stated in the following

Theorem 5.2 (*Interpolation in two variables*)
Let $\{ u_j^n \}$ $(j \in \mathbb{Z}\, , \, n \in \mathbb{Z}\, ; \, u_n \in \mathbb{R})$ be an arbitrary doubly-indexed infinite sequence of real numbers. Among all the smooth interpolants $u(x,t)$ satisfying :

$$\forall j \in \mathbb{Z}\, , \, \forall n \in \mathbb{Z} \, : \; u(j,n) = u_j^n \tag{126}$$

uncountably many are entire in both x and t separately, including the functions $u_{\lambda,\lambda'}(x,t)$ given by (125) for all $(\lambda, \lambda') \in \mathbb{R}_+^* \times \mathbb{R}_+^*$.

Remark : An alternate bounding entire function $\mu(x,t)$ can be constructed as follows. Choose an integer $q \geq 1$ (arbitrary for theoretical purpose, but preferably large for a sharper bound, see below), and consider the following real sequence :

$$U_k = \max_{\substack{(\ell,\eta)\in\mathbb{Z}^2/ \\ |\ell|^{2q}\leq k\, , \; |\eta|^{2q}\leq k}} |u_\ell^\eta| \qquad (k \in \mathbb{N}) \tag{127}$$

This sequence is positive and monotone increasing. It includes repeated elements; for example if $q = 1$: $U_1 = U_2 = U_3$, $U_4 = U_5 = U_6 = U_7 = U_8$, etc. Exceptionally, its first few elements may be equal to zero; if this is the case, replace them by the first (and smallest) strictly positive element of the sequence, and denote by $\{ U'_k \}$ the augmented sequence. By virtue of Theorem 3.1, uncountably many entire functions $\mu(s)$ of the single real variable s are such that :

$$\forall k \in \mathbb{N} \, : \; \mu(k) \geq U'_k \tag{128}$$

Then let $(j,n) \in \mathbb{Z} \times \mathbb{Z}$ be an arbitrary pair of indices, and observe that :

$$|u_j^n| \leq U_{\max(j^{2q},n^{2q})} \leq U_{j^{2q}+n^{2q}} \leq U'_{j^{2q}+n^{2q}} \leq \mu(j^{2q}+n^{2q}) \tag{129}$$

Then, in (125), replace $\mu(x,t)$ and $\mu(j,n)$ by $\mu(x^{2q}+t^{2q})$ and $\mu(j^{2q}+n^{2q})$ respectively.

In the case of a sequence $\{ u_j^n \}$ that grows radially as $|u_j^n| = f(j^2+n^2)$ for some strictly-positive monotone-increasing function f, $|u_j^n|$ achieves its maximum in the square $[-k^{\frac{1}{2q}}, k^{\frac{1}{2q}}]^2$ at $j = n \sim k^{\frac{1}{2q}}$ (for large k), giving $|u_j^n| = U_k = U'_k \sim f(2k^{\frac{1}{q}})$, while the bound over this quantity is taken to be in (125) : $\mu(j^{2q}+n^{2q}) = U_{j^{2q}+n^{2q}} = U_{2k} \sim f(2(2k)^{\frac{1}{q}}$ (in case $\mu(s)$ interpolates the sequence $\{ U_k \}$). Hence, larger values of q correspond to sharper bounds on $|u_j^n|$ (since $2^{\frac{1}{q}}$ is closer to 1).

6 Remarks, examples, concluding open questions

6.1 Behavior of the bounds as the parameter λ varies

Given a bounded sequence, $\{ u_n \}$, the constructive proof of the basic lemma 4.1 provides us, in (78), with an uncountable set of uniformly-bounded, entire interpolants $\{ u_\lambda \}$ ($\lambda \in \mathbb{R}_+^*$), admitting uniformly bounded derivatives. However it is *not said* that the sequence of upper bounds on the derivatives of increasing order is itself bounded. Besides, the parameter λ can assume any strictly positive value, but the behavior of these bounds as this parameter varies is uncertain. In this subsection, we consider the case of the specific sequence

$$u_n = \begin{cases} (-1)^n & \text{for } n < 0 \\ 0 & \text{for } n = 0 \\ (-1)^{n-1} & \text{for } n > 0 \end{cases} \tag{130}$$

for which we show that the sup-norm of the derivative u'_λ tends to infinity in both limits $\lambda \to 0^+$ and $\lambda \to \infty$.

Limit of $\|u'_\lambda\|_\infty$ as $\lambda \to 0^+$: Examine the value

$$\begin{aligned} F(\lambda) &= u_\lambda(\tfrac{1}{2}) \\ &= \sum_{n=1}^{\infty} \left((-1)^n \frac{\sin \pi(\frac{1}{2}+n)}{\pi(\frac{1}{2}+n)}\, e^{-\lambda(\frac{1}{2}+n)^2} + (-1)^{n-1} \frac{\sin \pi(\frac{1}{2}-n)}{\pi(\frac{1}{2}-n)}\, e^{-\lambda(\frac{1}{2}-n)^2} \right) \\ &= \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\frac{e^{-\lambda(n+\frac{1}{2})^2}}{n+\frac{1}{2}} + \frac{e^{-\lambda(n-\frac{1}{2})^2}}{n-\frac{1}{2}} \right) \\ &= \frac{2}{\pi}\, e^{-\frac{\lambda}{4}} + \frac{2}{\pi}\, G(\lambda) \end{aligned} \tag{131}$$

where

$$G(\lambda) = \sum_{n=1}^{\infty} \frac{e^{-\lambda(n+\frac{1}{2})^2}}{n+\frac{1}{2}} \tag{132}$$

Clearly,

$$G(\lambda) > \int_{\frac{3}{2}}^{\infty} \frac{e^{-\lambda x^2}}{x}\, dx = \int_{\frac{3\sqrt{\lambda}}{2}}^{\infty} \frac{e^{-s^2}}{s}\, ds > \int_{\frac{3\sqrt{\lambda}}{2}}^{1} \frac{e^{-s^2}}{s}\, ds > \frac{1}{e} \int_{\frac{3\sqrt{\lambda}}{2}}^{1} \frac{ds}{s} = \frac{1}{e}\, \ln \frac{2}{3\sqrt{\lambda}} \tag{133}$$

Consequently :

$$\lim_{\lambda \to 0^+} u_\lambda(\tfrac{1}{2}) = \infty \tag{134}$$

and since $u_\lambda(0) = u_0 = 0$, it follows that :

$$\lim_{\lambda \to 0^+} \| u'_\lambda \|_\infty = \infty \tag{135}$$

Limit of $\|u'_\lambda\|_\infty$ as $\lambda \to \infty$: Since $u_0 = 0$, $u_n = \pm 1$ for all $n \neq 0$, and $|\varphi(x-n)| \leq 1$ for all (x, n), it follows from the expression of $u_\lambda(x)$ in (78) that for any fixed x :

$$|u_\lambda(x)| \leq \sum_{n \in \mathbb{Z}^*} e^{-\lambda(x-n)^2} \tag{136}$$

Let x be fixed in $]0,1[$. Then :

$$|u_\lambda(x)| \leq \sum_{n=-1}^{-\infty} e^{-\lambda n^2} + e^{-\lambda(1-x)^2} + \sum_{n=2}^{\infty} e^{-\lambda(n-1)^2} = e^{-\lambda(1-x)^2} + 2 \sum_{n=1}^{\infty} e^{-\lambda n^2} \tag{137}$$

Therefore :

$$| u_\lambda(x) | \le e^{-\lambda(1-x)^2} + 2 \sum_{n=1}^{\infty} e^{-\lambda n} = e^{-\lambda(1-x)^2} + \frac{2e^{-\lambda}}{1-e^{-\lambda}} \to 0 \text{ as } \lambda \to \infty \tag{138}$$

and since $u_\lambda(1) = u_1 = 1$, it follows that :

$$\lim_{\lambda \to \infty} \| u'_\lambda \|_\infty = \infty \tag{139}$$

Conclusion : Figures 1-7 illustrate the behavior of the interpolants $u_\lambda(x)$ of (78) and $u_v(x)$ of (94) in the particular case of the sequence $\{ u_n \}$ given by (130) as the parameter λ increases from 0.001 to 1000, and the parameter ϵ is set to 0.01. We note that a too small value of the parameter λ produces an interpolant $u_\lambda(x)$ having large oscillations between the bounded interpolated values. The interpolant $u_v(x)$ remains within the bounds of the sequence : as expected the interpolated points correspond to local extrema, but this is realized at the expense of the appearence of unnatural oscillations between these extrema. Inversely, for an excessively large value of λ, the exponential factor seems to eliminate all possible spurious oscillations in $u_\lambda(x)$, but both interpolants nearly converge to the super-imposition of a sequence of isolated peaks, which, in a sense, is somewhat contradictory with the notion of an entire function, and consequently, the derivative becomes large in the neighborhood of the interpolation points. For a practical interpolation, an intermediate value of λ is optimal; for example, both interpolants are very "regular" for $\lambda = 1$.

6.2 A few open questions

We motivated our interest for "regular" interpolants of infinite sequences, by the necessity to construct a function $\tilde{u}(x,t)$ that coincides at the gridpoints of a regular mesh with the data produced by a time-integration scheme. This function is used in the Modified Equation Approach to identify the coefficients P_k of (21) and (22). If the function $\tilde{u}(x,t)$ can be constructed entire in x and t, we note that the existence, unicity and exact regularity of a function $v(x,t)$ solution of (22) is less clear.

To conclude, the results in this report lead us to raise a number of interesting related interpolation problems, mostly of theoretical interest, whose solutions are not known by us, in particular the following questions :

– When one interpolates a *finite* set of data, polynomial interpolation usually easily permits to match function values as well as derivatives (Lagrange or Hermite interpolation), and uniqueness of the interpolant is often the consequence of imposing an additional least-degree-type condition. In the case of an *infinite* (countable) set of data, and entire interpolants, which type of condition could guarantee uniqueness in a sensible way ? (Minimal $\|u'\|$? – Not applicable to a sequence with unbounded variations !)

– Furthermore, under the hypothesis of a bounded sequence $\{ u_n \}$, or a weaker one, can one define, possibly identify explicitly an entire interpolant with least norm – sup-norm or another norm – of the derivative u' ?

– Can one define a "high-frequency filter" that applied to a known entire interpolant would produce another entire interpolant, less oscillatory in some sense ?

– Could the pseudo-time integration of the heat equation

$$u_t = \epsilon(x)\, u_{xx} \ , \ \ u(x,0) = u_\lambda(x) \tag{140}$$

with $\epsilon(x) = 0$ for $x \in \mathbb{Z}$ and >0 otherwise (e.g. $\epsilon(x) = \epsilon \sin^2 \pi x$), serve such purpose ?

– For a monotone sequence $\{ u_n \}$, exists there always a monotone interpolant that is entire ?

- Concerning Hermitian interpolation, if bounds are known on the sequence and the first p sequences of divided differences, can entire interpolants $u(x)$ be constructed so that $u(x)$, $u'(x)$, $\dots u^{(p)}(x)$ admit uniform bounds over the real line in relation with these known bounds respectively ?

- Could the interpolants constructed in this report, or variants, be used in a practical context to form a special basis of some value for the numerical solution of a P.D.E. ?

7 Bibliography

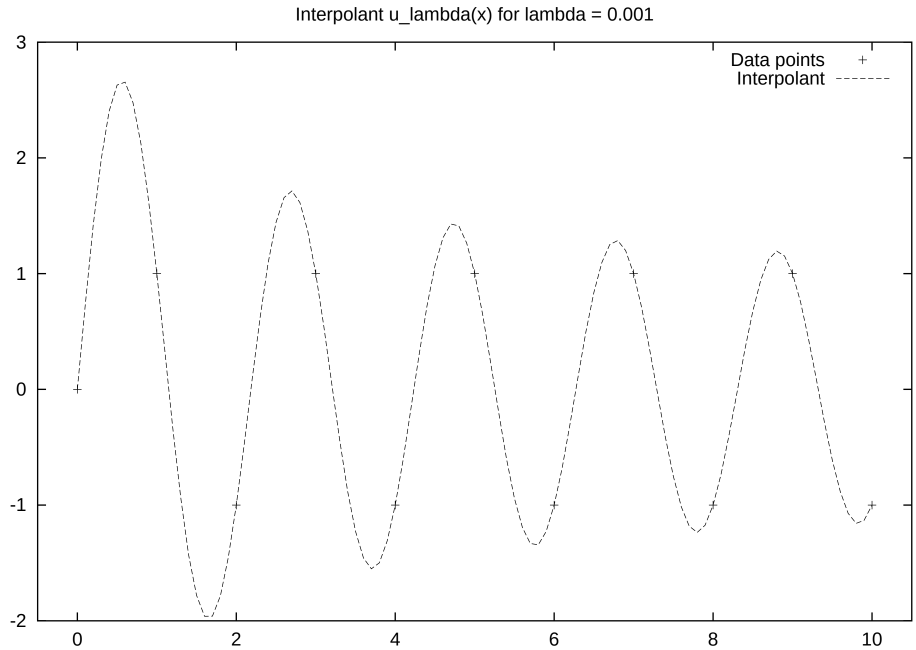
[1] R.F. Warming and B.J. Hyett. The modified equation approach to the stability and accuracy analysis of finite-difference methods. *J. Comp. Phys.*, 14:159–179, 1974.

[2] A. Lerat and R. Peyret. Propriétés dispersives et dissipatives d'une classe de schémas aux différences pour les systèmes hyperboliques nonlinéaires. *La Recherche Aérospatiale*, 1975-2:61–79, 1975.

[3] Sin-Chung Chang. A critical analysis of the modified equation technique of warming and hyett. *J. Comp. Phys.*, 86:107–126, 1990.

[4] Jr. R.P. Boas. *Entire Functions.* Academic Press, New York, 1954.

(a) $u_\lambda(x)$



(b) $u_v(x)$

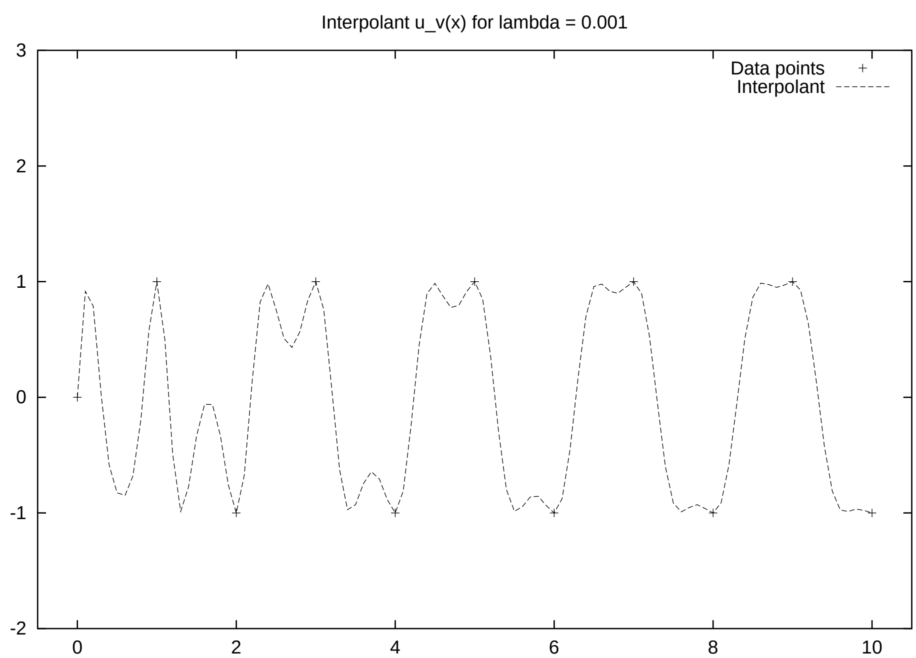
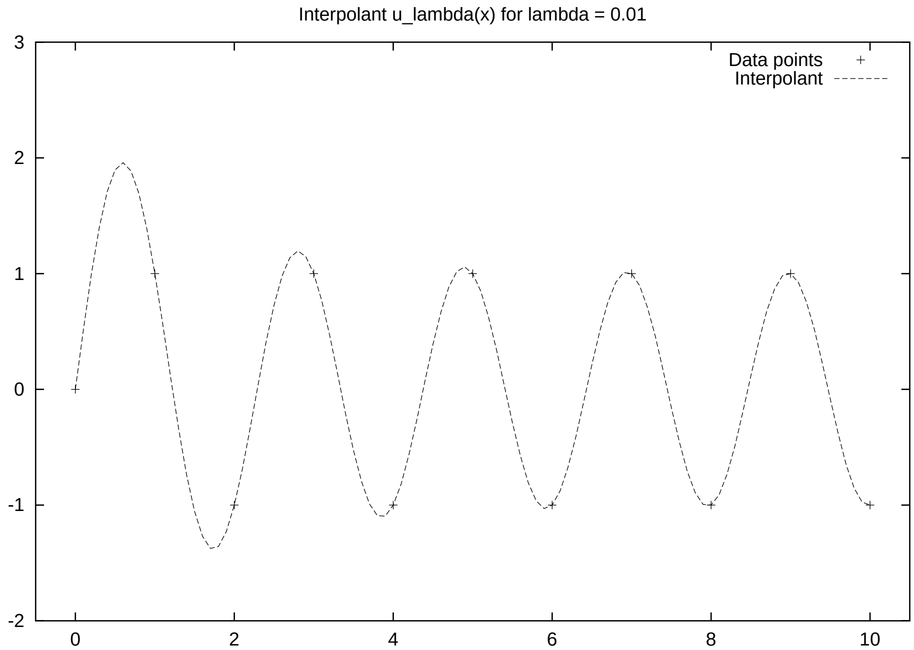


Figure 1: Behavior of the interpolants $u_\lambda(x)$ of (78) and $u_v(x)$ of (94) in the particular case of the sequence $\{ u_n \}$ given by (130); $\lambda = 0.001$, $\epsilon = 0.01$.

(a) $u_\lambda(x)$



(b) $u_v(x)$

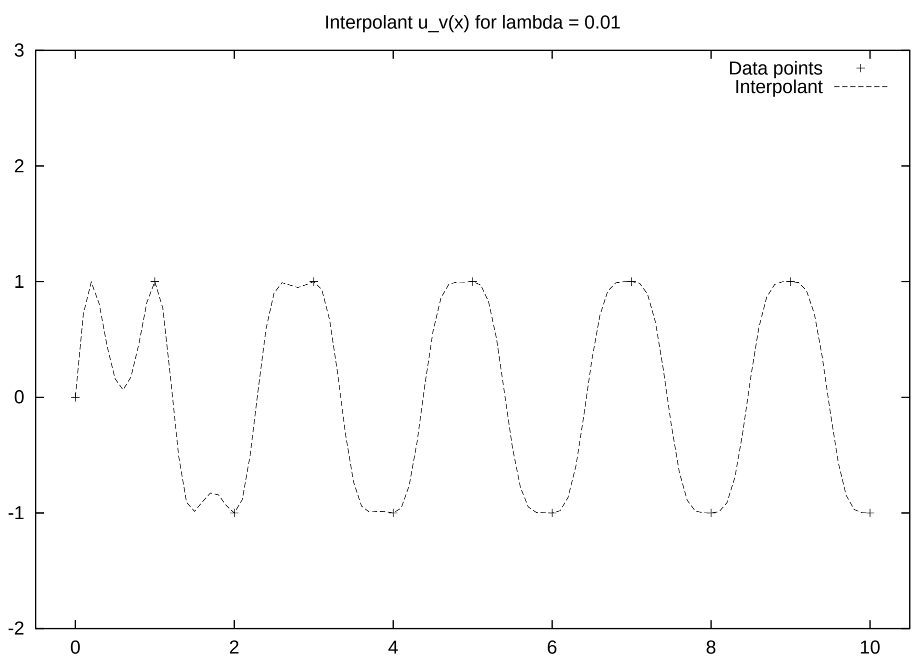
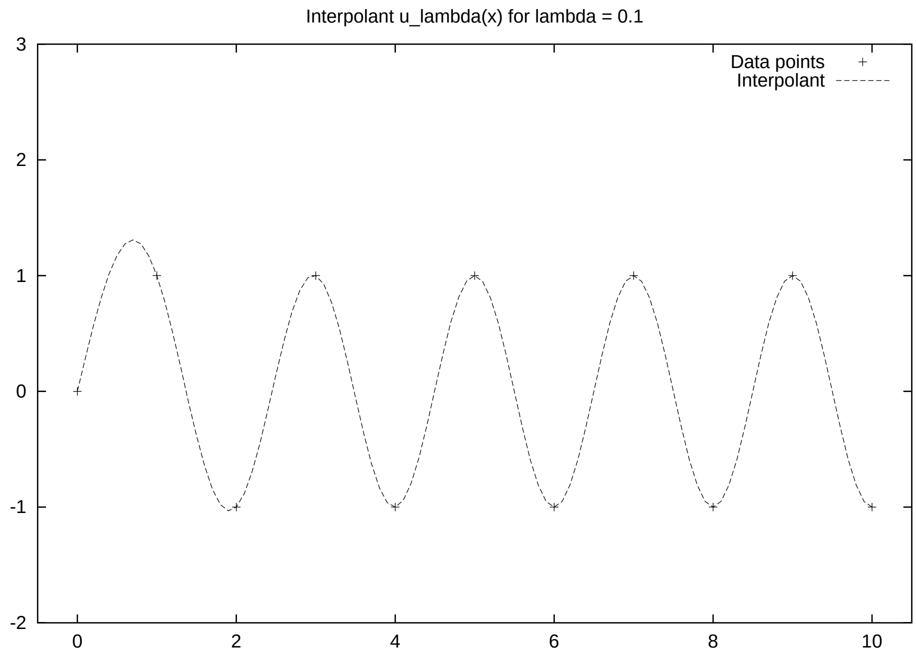


Figure 2: Behavior of the interpolants $u_\lambda(x)$ of (78) and $u_v(x)$ of (94) in the particular case of the sequence $\{ u_n \}$ given by (130); $\lambda = 0.01$, $\epsilon = 0.01$.

(a) $u_\lambda(x)$



(b) $u_v(x)$

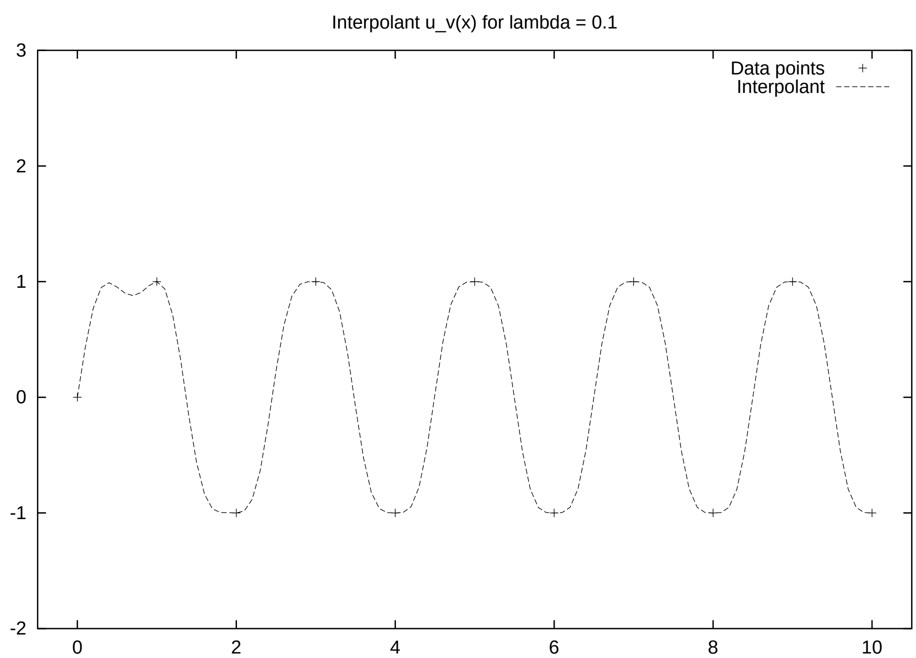
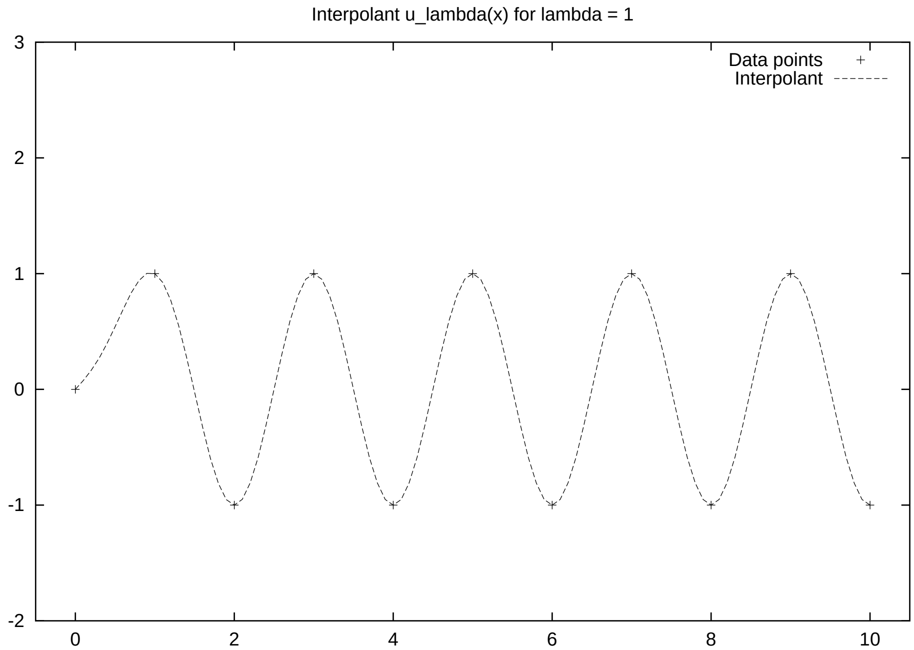


Figure 3: Behavior of the interpolants $u_\lambda(x)$ of (78) and $u_v(x)$ of (94) in the particular case of the sequence $\{ u_n \}$ given by (130); $\lambda = 0.1$, $\epsilon = 0.01$.

(a) $u_\lambda(x)$



(b) $u_v(x)$

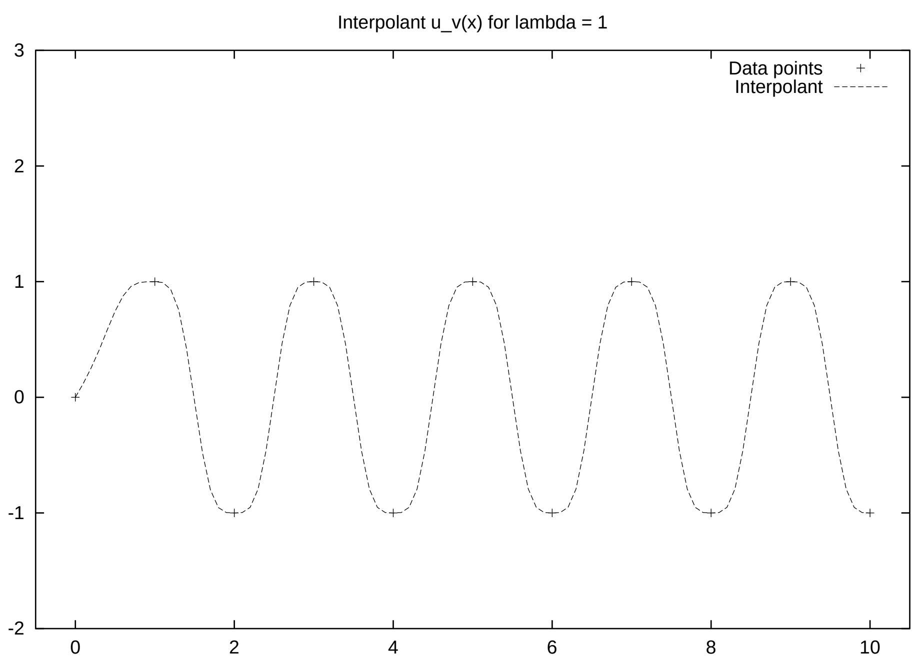
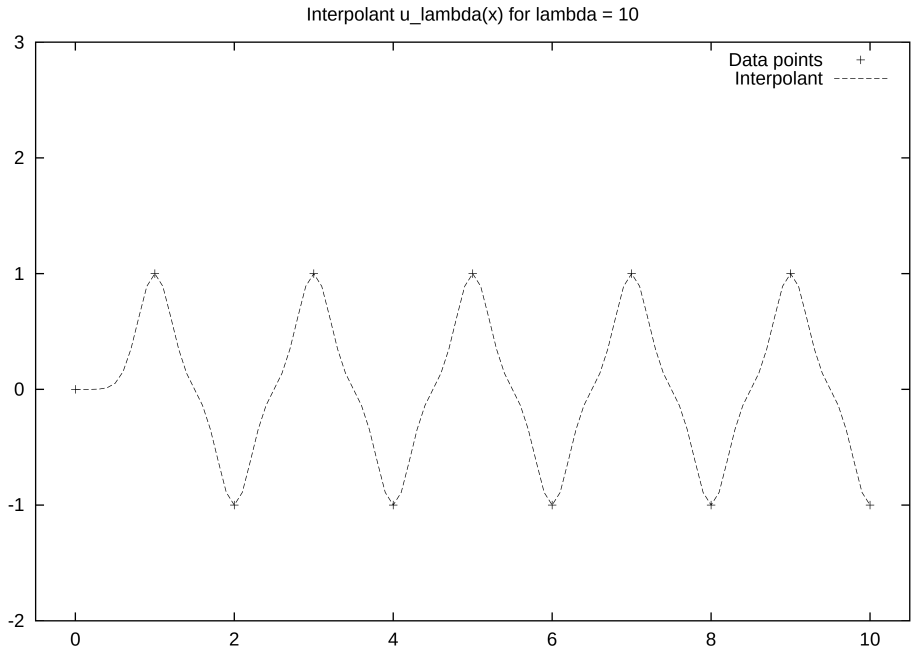


Figure 4: Behavior of the interpolants $u_\lambda(x)$ of (78) and $u_v(x)$ of (94) in the particular case of the sequence $\{\, u_n \,\}$ given by (130); $\lambda = 1$, $\epsilon = 0.01$.

(a) $u_\lambda(x)$



(b) $u_v(x)$

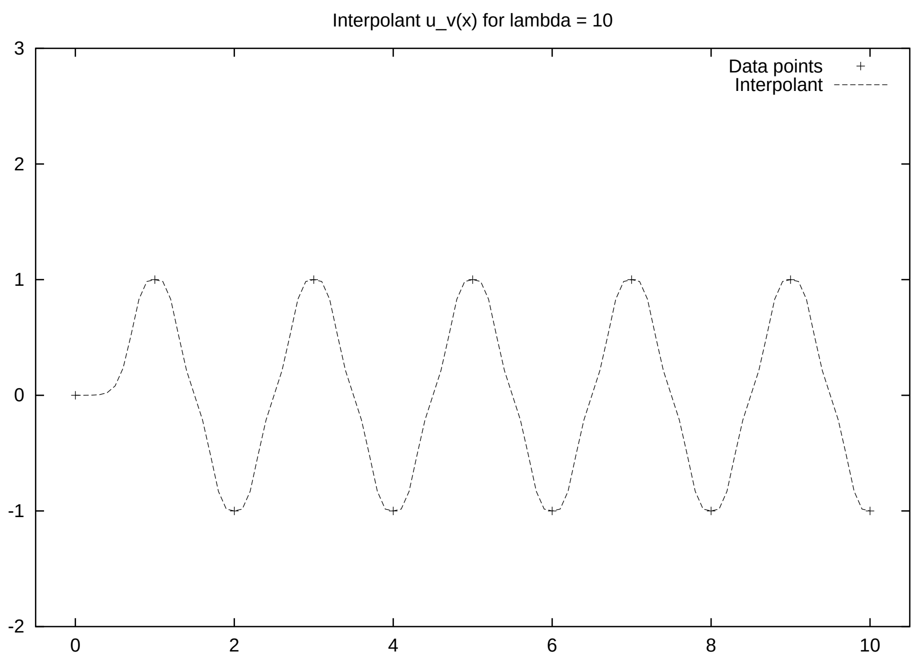
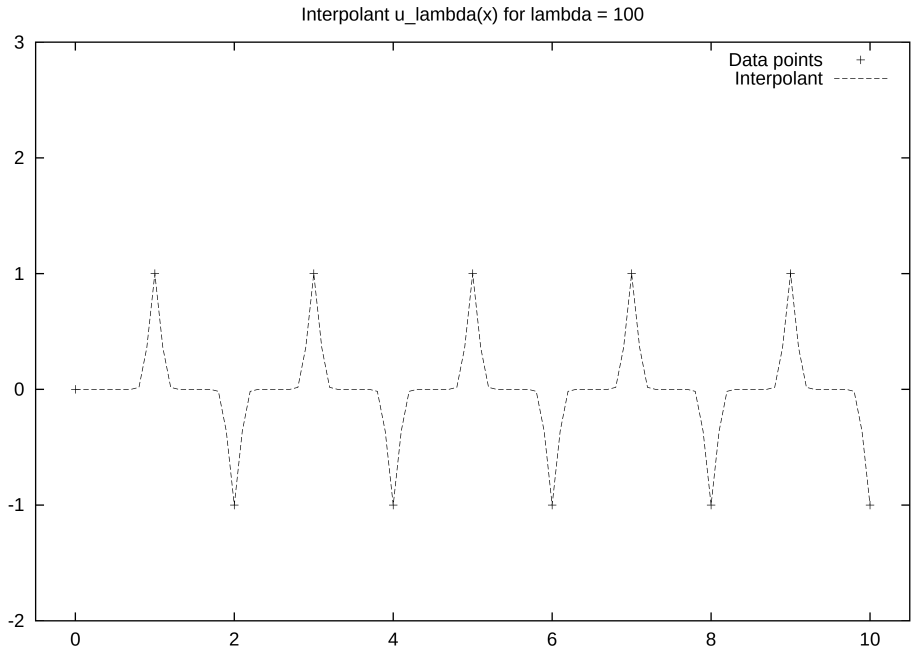


Figure 5: Behavior of the interpolants $u_\lambda(x)$ of (78) and $u_v(x)$ of (94) in the particular case of the sequence $\{ u_n \}$ given by (130); $\lambda = 10$, $\epsilon = 0.01$.

(a) $u_\lambda(x)$



(b) $u_v(x)$

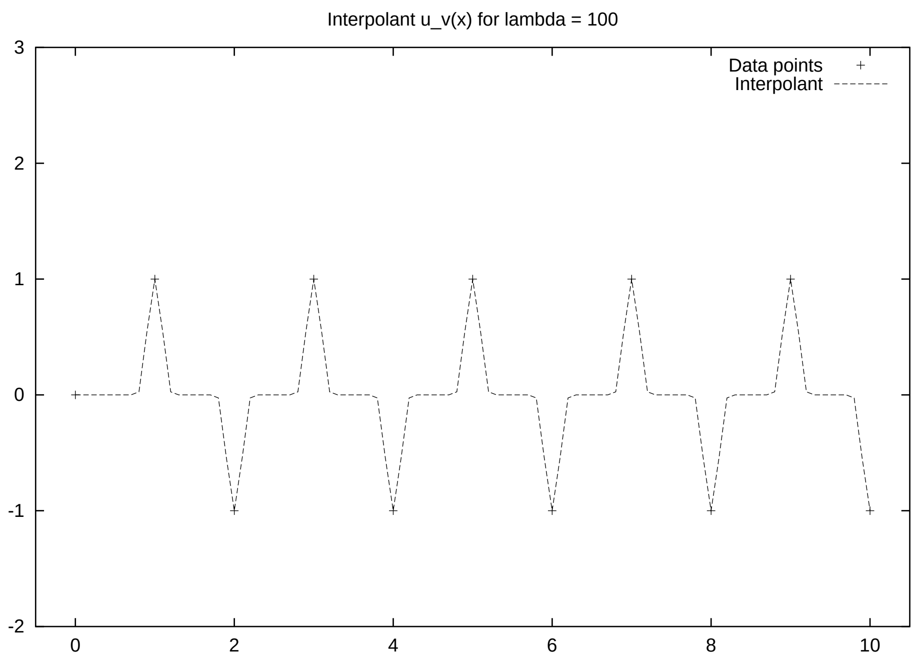
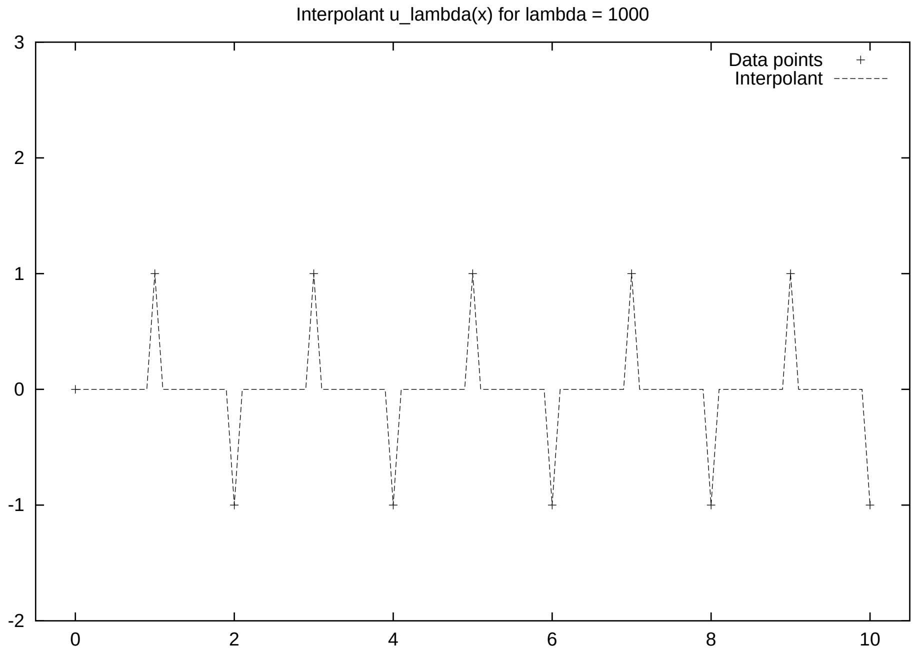


Figure 6: Behavior of the interpolants $u_\lambda(x)$ of (78) and $u_v(x)$ of (94) in the particular case of the sequence $\{ u_n \}$ given by (130); $\lambda = 100$, $\epsilon = 0.01$.

(a) $u_\lambda(x)$



(b) $u_v(x)$

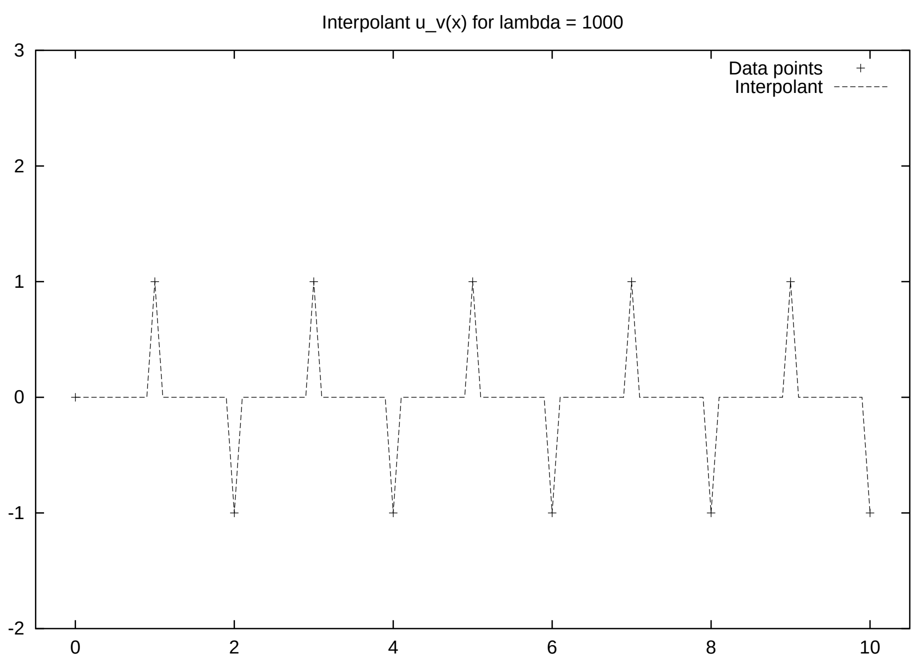


Figure 7: Behavior of the interpolants $u_\lambda(x)$ of (78) and $u_v(x)$ of (94) in the particular case of the sequence $\{ u_n \}$ given by (130); $\lambda = 1000$, $\epsilon = 0.01$.

A Appendix : Glossary of results

For convenience, we collect in this appendix the results of our study.

Theorem 3.1 [*Bounding an infinite sequence by the values of an entire function*]
Let $\{u_n\}$ ($n \in \mathbb{Z}$; $u_n \in \mathbb{R}$) be an arbitrary infinite sequence of real numbers. Uncountably many real-valued entire functions $\mu(x)$ of the real variable x are such that :

$$\forall n \in \mathbb{Z} : \ |u_n| \leq \mu(n) \tag{27}$$

Lemma 4.1 *[Basic Lemma for bounded sequences]*
Let $\{ u_n \}$ ($n \in \mathbb{Z}$; $u_n \in \mathbb{R}$) be a bounded infinite sequence of real numbers. Among all the smooth interpolants $u(x)$ satisfying :

$$\forall n \in \mathbb{Z} : \ u(n) = u_n \tag{59}$$

uncountably many are entire, uniformly bounded, and admit derivatives that are uniformly bounded over the real line.

Corollary 4.1 *[On bounded interpolants of a bounded sequence]*
Let $\{ u_n \}$ ($n \in \mathbb{Z}$; $u_n \in \mathbb{R}$) be a bounded infinite sequence of real numbers satisfying :

$$\forall n \in \mathbb{Z} : \ -1 \leq u_n \leq 1 \tag{87}$$

Among all the smooth interpolants $u(x)$ satisfying :

$$\forall n \in \mathbb{Z} : \ u(n) = u_n \tag{88}$$

uncountably many are entire, satisfy

$$\forall x \in \mathbb{R} \setminus \mathbb{Z} : \ -1 < u(x) < 1 \tag{89}$$

and admit derivatives that are uniformly bounded over the real line.

Theorem 4.1 *[Interpolating a general infinite sequence]*
Let $\{ u_n \}$ ($n \in \mathbb{Z}$; $u_n \in \mathbb{R}$) be an arbitrary infinite sequence of real numbers. Uncountably many pairs of entire functions $\Big(a(x)\,,\ b(x)\Big)$ are such that :

$$\forall n \in \mathbb{Z} : \ \min\Big(a(n)\,,\ b(n)\Big) \leq u_n \leq \max\Big(a(n)\,,\ b(n)\Big) \tag{103}$$

Additionally, for any such pair for which $a(x)$ and $b(x)$ are not identical, among all the smooth interpolants $u(x)$ satisfying :

$$\forall n \in \mathbb{Z} : \ u(n) = u_n \tag{104}$$

uncountably many are of the form :

$$u(x) = \frac{b(x) + a(x)}{2} + \frac{b(x) - a(x)}{2}\,\theta(x) \tag{105}$$

where the function $\theta(x)$ is entire, uniformly bounded, admits uniformly bounded derivatives over the real line, and satisfies :

$$\begin{cases} \forall n \in \mathbb{Z} : \ \theta(n) = \theta_n \\ \forall x \in \mathbb{R} \setminus \mathbb{Z} : \ -1 < \theta(x) < 1 \end{cases} \tag{106}$$

where the sequence $\{\,\theta_n\,\}$ *has been defined by :*

$$\theta_n = \begin{cases} \left[u_n - \dfrac{b(n)+a(n)}{2} \right] \Big/ \left[\dfrac{b(n)-a(n)}{2} \right] & \textit{whenever } a(n) \neq b(n) \\ \textit{any number in [-1,1], otherwise} \end{cases} \tag{107}$$

Lemma 5.1 *[Matching the derivative of order p]*
Let p *be a strictly-positive integer,* $\theta_p(x)$ *a function of class* $\mathsf{C}^\infty(\mathbb{R})$*, and :*

$$v_p(x) = \Big(\sin\,\pi x\Big)^p\,\theta_p(x) \tag{109}$$

Then :

$$\forall n \in \mathbb{Z}\ :\ \begin{cases} v_p(n) = v_p'(n) = \cdots = v_p^{(p-1)}(n) = 0 \\ v_p^{(p)}(n) = (-1)^{np}\,\pi^p\,p!\,\theta_p(n) \end{cases} \tag{110}$$

Theorem 5.1 *[Hermitian interpolation]*
Let p *be a positive integer, and* $\{\,u_n^k\,\}$ *(*$n \in \mathbb{Z}\,,\ k \in \mathbb{N},\ k \leq p\,,\ u_n^k \in \mathbb{R}$*) an arbitrary infinite doubly-indexed sequence of real numbers. Among all the smooth interpolants* $u(x)$ *satisfying :*

$$\forall n \in \mathbb{Z}\,,\ \forall k \in \mathbb{N} \textit{ such that } k \leq p\ :\ u^{(k)}(n) = u_n^k \tag{116}$$

uncountably many are entire.

Theorem 5.2 *[Interpolation in two variables]*
Let $\{\,u_j^n\,\}$ *(*$j \in \mathbb{Z}\,,\ n \in \mathbb{Z}\,;\ u_n \in \mathbb{R}$*) be an arbitrary doubly-indexed infinite sequence of real numbers. Among all the smooth interpolants* $u(x,t)$ *satisfying :*

$$\forall j \in \mathbb{Z}\,,\ \forall n \in \mathbb{Z}\ :\ u(j,n) = u_j^n \tag{126}$$

uncountably many are entire in both x *and* t *separately, including the functions* $u_{\lambda,\lambda'}(x,t)$ *given by (125) for all* $(\lambda,\lambda') \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ *.*

Contents

Epilogue

I have often wondered if my year in the French military served any useful purpose. Now I know that it served in preparing this report!

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